Games on Networks with Community Structure: Existence, Uniqueness and Stability of Equilibria

Kun Jin, Mohammad Mahdi Khalili and Mingyan Liu

***Abstract*— We study games with nonlinear best response functions played on a network consisting of disjoint communities. Prior works on network games have identified conditions to guarantee the uniqueness and stability of Nash equilibria in a network without any community structure. In this paper we are interested in accomplishing the same for networks with a community structure; it turns out that these conditions are much easier to verify with certain community structures. Specifically, we consider multipartite graphs and show that the uniqueness and stability of Nash equilibria are related to matrices which are potentially much lower in dimension, on the order of the number of communities, compared to same-size networks without a multipartite structure, in which case such matrices have a dimension the size of the network. We further introduce a new notion of degree centrality to measure the importance and influence of a community in such a network. We show that this notion enables us to find new conditions for uniqueness and stability of Nash equilibria.**

I. INTRODUCTION

Interaction of strategic agents and their decision making process are considered as a network game. In this type of games, the utility of an agent depends on his own effort/decision as well as the effort of other agents in his neighborhood. Specially, the effort of an agent can be a substitute or complement for his neighbors. If the effort of an agent is a substitute (complement) to his neighbors, then an increase in effort of an agent increases (decreases) the utility of the neighbors. Various models have been proposed and studied as a network game such as provision of public goods [1], [2], [3], Interdependent Security Games [4], [5], and financial markets [6].

A common line of research in this literature is to study the effect of network structure on the existence, uniqueness and stability of the equilibrium (see, e.g., [7] for a survey on network games). Network games with linear best response functions have been studied in [8], [9]. Bramoulle *et al.* [8] uncovers the importance of the lowest eigenvalue of adjacency matrix of the network and shows that the uniqueness and stability of the equilibrium depend on this eigenvalue. [9] shows that if the adjacency matrix of the network is strictly diagonally dominant, then the equilibrium is unique.

Network games with nonlinear best-response functions have been studied in [1], [10], [11], [12], [13]. Allouch in [1] introduces a sufficient condition for the uniqueness of Nash equilibrium called *network normality* which imposes lower

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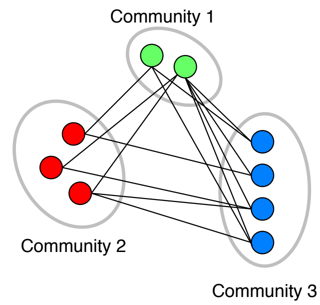

Fig. 1. A multipartite network with 3 communities.

and upper bound on derivative of Engel curves. Acemoglu *et al.* [10] consider a network game with *idiosyncratic* shocks and show that if the best response mapping is either a contraction with a Lipschitz constant smaller than one or a bounded non-expansive mapping, then the game has a unique Nash equilibrium. Zhou *et al.* [11] establish a connection between nonlinear complementary problem (NCP) and network games and use the existing results in NCP literature to find sufficient conditions for uniqueness of Nash equilibrium. Recent work by Parise and Ozdaglar [12] identifies conditions on the underlying network using a variation inequality framework to guarantee existence and uniqueness of Nash equilibrium. Lastly, [13] shows that the uniqueness and stability of the Nash equilibrium can be determined by the lowest eigenvalue of matrices which depend on the slope of the agents' interaction functions and intensity of their interactions.

In all of the aforementioned literature, the underlying network is given by a generic adjacency matrix. By contract, in this paper, we are interested in structured networks, in particular, networks with communities. Specifically, we shall consider a network with disjoint communities forming a multipartite graph/network (see Figure 1 for an example of a network with three communities and nine agents/players), where an agent is connected to some or all other communities (through agent(s) belonging to that community), but not to anyone within its own community. An example of this type of network is a network of internet service providers (ISPs), security product/software vendors, and end user/customers, where a community in this context is formed based on the agent's type. In this case, the connections symbolize dependent or supply chain relationships, while agents of the same type do not depend on each other.

Specifically, we will similarly consider a network game with nonlinear best-response functions as in [1], [10], [11],

[12], [13], but over a multipartite graph. Our goal is to understand how the existence of this type of community structure affect the resulting equilibrium analysis. Our main results are summarized as follows.

1) While prior work, e.g., [13] provides sufficient conditions under which the Nash equilibrium in a network with a general structure is unique and stable, we show that the existence of a multipartite community structure enables us to find conditions which are simpler and *easier* to verify. In particular, while the computational effort of the verification for a structureless network depends on the size of the network (total number of agents), for a network with multipartite structure, the verification now depends only on the number of communities, which is typically much smaller than the size of the network.
2) In addition to the uniqueness and stability of the Nash equilibrium, we also introduce a new notion of centrality which can be considered as a generalization of degree centrality. This new centrality definition measures the influence/importance of a community in the network. We use this notion of centrality to identify new conditions for the uniqueness and stability of Nash equilibrium.

It is worth noting that bipartite and multipartite graphs/networks are widely studied in a variety of games beyond the type of network games considered in this study. Bipartite graphs/networks are studied in network security games with attackers and a defender [14], matching games [15], and network formation game in bipartite exchange economies [16]. In [17], the fair cost allocations game was analyzed on bipartite and complete multipartite graphs.

The organization of the remainder of the paper is as follows. In Section II we introduce our game model with community structure. We present our results on the existence and uniqueness of Nash equilibria in Section III, followed by results on the stability of Nash equilibria in Section IV. We present and discuss a generalized notion of degree centrality in Section V, and conclude the paper in Section VI.

II. Model and Preliminaries

A. Game Model

We consider a network with N agents divided into M different and disjoint communities represented by a directed graph $\mathscr{G} = (\mathscr{N}, \mathscr{E})$, where $\mathscr{N}$ is the set of nodes/agents and $\mathscr{E} \subset \mathscr{N} \times \mathscr{N}$ the set of edges. Let $C_i = \{a_i^{(1)}, a_i^{(2)}, \cdots, a_i^{(N_i)}\}$ denote community i where N_i is its size and $a_i^{(k)}$ denotes its k-th agent.

Agent $a_i^{(k)}$ exerts an effort level $x_i^{(k)} \geq 0$. $\boldsymbol{x}_i = [x_i^{(1)}, \cdots, x_i^{(N_i)}]^T$ denotes the effort vector of community i.

An edge weight $w_{ij}^{(k,r)}$ is a real number representing the type and strength of agent $a_j^{(r)}$'s influence on agent $a_i^{(k)}$; positive (negative) weights are considered as strategic substitutes (complements). More precisely, if $w_{ij}^{(k,r)}$ is positive (negative), then an increase in effort of agent $a_j^{(r)}$ increases (decreases) $a_i^{(k)}$'s utility due to the positive (negative) externality. Moreover, $w_{ij}^{(k,r)} = 0$ implies that agent $a_j^{(r)}$ does not influence $a_i^{(k)}$.

$W_{ij} = [w_{ij}^{(k,r)}]_{N_i \times N_j}$ is a weight matrix between communities i and j. The multipartite graph assumption translates into the property $W_{ii} = \mathbf{0}$, $\forall i$. An example of this type of network and its interpretation were given in the introduction.

We denote by $\mathscr{N}_{C_i}$ the set of indices of communities which are neighbors of community i, i.e., with whom i has some connection: $\mathscr{N}_{C_i} := \{j | W_{ij} \neq \mathbf{0}, j \neq i\}$.

We consider a family of games with the following best response function,

$$\begin{aligned} x_i^{(k)} &= \max\{f_{i,k}([\sum_{j=1}^{M} W_{ij} \cdot \boldsymbol{x}_j]_k), 0\}, \\ \forall i &\in \{1, 2, \cdots, M\},\ k \in \{1, 2, \cdots, N_i\}, \end{aligned} \tag{1}$$

where $[\boldsymbol{a}]_k$ is the kth element of vector $\boldsymbol{a}$, and the maximization is taken to make sure that effort $x_i^{(k)}$ remains non-negative and feasible. Here, $f_{i,k}(.) : \mathbb{R} \to \mathbb{R}$ is the interaction function between agent $a_i^{(k)}$ and the other agents. We will keep this function general and shall only assume $f_{i,k}(.)$ is continuously differentiable for any $i \in \{1, 2, \cdots, M\}$, $k \in \{1, 2, \cdots, N_i\}$. We provide two examples of utility function which give us the best response function in form (1).

Example 1 (quadratic utility function): Let agent k in community i have the following utility function,

$$\begin{aligned} u_i^{(k)}(\boldsymbol{x}) = &\theta_i^{(k)} \cdot x_i^{(k)} - \frac{1}{2}(x_i^{(k)})^2 \\ &+ x_i^{(k)} \cdot f_{i,k}\left(\sum_{j=1}^{M} \sum_{r=1}^{N_j} w_{ij}^{(k,r)} \cdot x_j^{(r)}\right) \end{aligned} \tag{2}$$

where $\theta_i^{(k)}$ is a fixed positive value. By the first order condition, the best response function is given by,

$$\begin{aligned} x_i^{(k)} &= \max\left\{\theta_i^{(k)} + f_{i,k}\left(\sum_{j=1}^{M} \sum_{r=1}^{N_j} w_{ij}^{(k,r)} \cdot x_j^{(r)}\right), 0\right\} \\ &= \max\left\{\theta_i^{(k)} + f_{i,k}([\sum_{j=1}^{M} W_{ij} \cdot \boldsymbol{x}_j]_k), 0\right\} \end{aligned} \tag{3}$$

This type of utility function has been studied for network games [18], [19].

Example 2: Let agent k in community i have the following utility function:

$$u_i^{(k)}(\boldsymbol{x}) = V\left(x_i^{(k)} + f_{i,k}\left(\sum_{j=1}^{M} \sum_{r=1}^{N_j} w_{ij}^{(k,r)} \cdot x_j^{(r)}\right)\right) - c_i^{(k)} \cdot x_i^{(k)} \tag{4}$$

where, $V(.) : \mathbb{R} \to \mathbb{R}$ is a real valued strictly increasing and strictly concave function and $c_i^{(k)}$ is the cost per unit of effort for agent $a_i^{(k)}$. Let $q_i^{(k)}$ be the effort level such that $V'(q_i^{(k)}) = c_i^{(k)}$. Then, by the first order condition, the best response function of agent $a_i^{(k)}$ is given by,

$$\begin{aligned} x_i^{(k)} &= \max\left\{q_i^{(k)} - f_{i,k}\left(\sum_{j=1}^{M}\sum_{r=1}^{N_j} w_{ij}^{(k,r)} \cdot x_j^{(r)}\right), 0\right\} \\ &= \max\left\{q_i^{(k)} - f_{i,k}([\sum_{j=1}^{M} W_{ij} \cdot \boldsymbol{x}_j]_k), 0\right\} \end{aligned} \quad (5)$$

This type of function has been studied in public good provision games and network games [11], [8], [20].

For convenience, we define $f_i = [f_{i,1}, f_{i,2}, \cdots, f_{i,N_i}]^T$. Therefore, the best response functions of the agents in community i are given by,

$$\boldsymbol{x}_i = \max\{f_i(\sum_{j=1}^{M} W_{ij} \cdot \boldsymbol{x}_j), \boldsymbol{0}\},\ \forall i \in \{1, 2, \cdots, M\} \quad (6)$$

where,

$$\max\{[a_1, a_2, \cdots, a_{N_i}]^T, \boldsymbol{0}\} = [\max\{a_1, 0\}, \cdots, \max\{a_{N_i}, 0\}]^T.$$

We are interested in characterizing the Nash equilibrium of the game induced by interactions between strategic agents in a network with the community structure. By definition, the Nash equilibrium is the fixed point of best-response mappings. Let $\boldsymbol{x}^* = [\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \cdots, \boldsymbol{x}_M^*]^T \in \mathbb{R}^N_{\geq 0}$ be the Nash equilibrium resulting from interaction of the agents. Then:

$$\begin{aligned} x_i^{(k)*} &= \max\{f_{i,k}([\sum_{j=1}^{M} W_{ij} \cdot \boldsymbol{x}_j^*]_k), 0\}, \\ \forall i &\in \{1, 2, \cdots, M\},\ k \in \{1, 2, \cdots, N_i\} \end{aligned} \quad (7)$$

or equivalently,

$$\boldsymbol{x}_i^* = \max\{f_i(\sum_{j \in \mathcal{N}_{C_i}} W_{ij} \cdot \boldsymbol{x}_j^*), \boldsymbol{0}\},\ \forall i \in \{1, 2, \cdots, M\}\ . \quad (8)$$

Moreover, (8) implies that $\boldsymbol{x}_i^*$ satisfies the following inequality:

$$(\boldsymbol{x}_i - \boldsymbol{x}_i^*)^T \cdot (\boldsymbol{x}_i^* - f_i(\sum_{j \in \mathcal{N}_{C_i}} W_{ij} \cdot \boldsymbol{x}_j^*)) \geq 0,\ \forall \boldsymbol{x}_i \in \mathbb{R}^{N_i}_{\geq 0}, \forall i \quad (9)$$

Next, we introduce Variational Inequality (VI) problem and its application in finding Nash equilibrium.

B. The Variational Inequality (VI) problem [21]

Variational Inequality is a class of mathematical problems and has application in optimization and fixed point problems. The variational inequality problem is defined as follows,

Definition 1: A varational inequality $VI(K, F)$ consists of a set $K \subset \mathbb{R}^N$ and a mapping $F : K \to \mathbb{R}^N$, and is the problem of finding a vector $\boldsymbol{x}^* \in K$ such that,

$$(\boldsymbol{x} - \boldsymbol{x}^*)^T F(\boldsymbol{x}^*) \geq 0,\ \forall \boldsymbol{x} \in K. \quad (10)$$

By (9) and (10), it is easy to see that finding Nash equilibrium of the game defined in Section II-A is equivalent to a variational inequality problem $VI(K, F(\boldsymbol{x}))$ where K is the action space, $F(\boldsymbol{x}) = [F_1(\boldsymbol{x}), F_2(\boldsymbol{x}), \cdots, F_M(\boldsymbol{x})]^T$, and $F_i(\boldsymbol{x}) = (\boldsymbol{x}_i - f_i(\sum_{j \in \mathcal{N}_{C_i}} W_{ij} \cdot \boldsymbol{x}_j))$.

III. Existence and Uniqueness

In this section we identify the conditions for the existence and uniqueness of Nash equilibrium based on the VI formulation of the game.

A. Existence of NE

We first explore the conditions under which Nash equilibrium exists. The game defined in Section II-A has at least one Nash equilibrium if the solution set of $VI(K, F(\boldsymbol{x}))$ is nonempty. The following theorem introduces a sufficient condition under which the VI problem has a nonempty solution set [22].

Theorem 1: ([22]) The $VI(K, F)$ has a nonempty and compact solution set, if the following conditions hold,
1. F is continuous on K.
2. K is nonempty, compact and convex.

Using Theorem 1, we have the following proposition.

Proposition 1: In $VI(K, F)$, given that F is continuous on K, if one of the following conditions is satisfied, then the Nash equilibrium exists.

1. Each agent has a finite effort budget and K is a Cartesian product of agents' action space. That is, $x_i^{(k)} \in K_i^{(k)} = [0, B_i^{(k)}], \forall i, k$ and $K = \prod_{i=1}^{M} \prod_{k=1}^{N_i} K_i^{(k)}$ where $B_i^{(k)}$ and $K_i^{(k)}$ are the budget and action space of agent $a_i^{(k)}$, respectively.

2. There is a finite and independent effort budget shared by each community, and based on the budget, we have a convex and compact action space K_i for community C_i. That is, $K_i = \{\boldsymbol{x}_i | \sum_{k=1}^{N_i} x_i^{(k)} \leq B_i,\ \boldsymbol{x}_i \succeq \boldsymbol{0}\}$ and $K = \prod_{i=1}^{M} K_i$, where B_i is the shared budget of C_i.

Proof: See Appendix I. ■

B. Uniqueness of NE

We next introduce sufficient conditions under which the network game defined in II-A has a unique solution. We begin by introducing the following definitions.

Definition 2: P-Matrix:
A matrix $A \in \mathbb{R}^{N \times N}$ is a P-matrix if every principal minor of A has a positive determinant.

Definition 3: A mapping $F : K \subseteq \mathbb{R}^N \to \mathbb{R}^N$, where K is nonempty, closed and convex and F is continuously differentiable on K, is

(a) Strongly monotone: if there exists a constant $a > 0$ such that

$$(\boldsymbol{x} - \boldsymbol{y})^T (F(\boldsymbol{y}) - F(\boldsymbol{y})) \geq a||\boldsymbol{x} - \boldsymbol{y}||_2^2,\ \forall \boldsymbol{x}, \boldsymbol{y} \in K \quad (11)$$

(b) A uniform block P-function w.r.t. the partition $K = K_1 \times K_2 \times \cdots \times K_M$: if there exists a constant $b > 0$ such that

$$\max_{i \in \mathbb{N}[1,M]} (\boldsymbol{x}_i - \boldsymbol{y}_i)^T [F_i(\boldsymbol{x}) - F_i(\boldsymbol{y})] \geq b||\boldsymbol{x} - \boldsymbol{y}||_2^2,\ \forall \boldsymbol{x}, \boldsymbol{y} \in K \quad (12)$$

By setting $b = a/M$, it is easy to see that strong monotonicity is a sufficient condition of uniform block P-condition. Parise and Ozdaglar [12] show that if $F(\boldsymbol{x})$ is strongly monotone or a uniform block P-function, then $VI(K, F)$ has the unique solution and the Nash equilibrium of network game corresponding to $VI(K,F)$ is unique.

Unfortunately, it is not easy to verify these conditions for function $F(\boldsymbol{x})$. Below we take advantage of the community structure and identify conditions for the uniqueness of Nash equilibrium that are simpler and easier to verify. To do so, we introduce matrix Υ^C as follows.

$$\beta_{ij}^C = \sup_{\boldsymbol{x}\in K} ||\nabla_j F_i(\boldsymbol{x})||_2, \ \forall i,j \in \mathbb{N}[1,M], \ i \neq j \quad (13)$$

$$\Upsilon^C = \begin{bmatrix} 1 & -\beta_{12}^C & \dots & -\beta_{1M}^C \\ -\beta_{21}^C & 1 & \dots & -\beta_{2M}^C \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_{M1}^C & -\beta_{M2}^C & \dots & 1 \end{bmatrix}, \quad (14)$$

where $\nabla_j F_i(\boldsymbol{x}) = \frac{\partial F_i(\boldsymbol{x})}{\partial \boldsymbol{x}_j} \in \mathbb{R}^{N_i \times N_j}$.

The following proposition identifies a condition for the uniqueness of Nash equilibrium.

Proposition 2: Consider the network game defined in Section II-A and assume the action space K is a Cartesian product $K = K_1 \times K_2 \times \cdots \times K_M$, where K_i is the action space of the agents in community C_i. If Υ^C is a P-matrix, then the network game has a unique Nash equilibrium.

Proof: (Sketch)

- First we show that if Υ^C is a P-matrix, then $F(\boldsymbol{x})$ is a uniform block P-function with respect to partition K.
- By [12], if $F(\boldsymbol{x})$ is a uniform block P-function with respect to partition K, then $VI(K,F)$ has unique solution which implies that the network game has a unique equilibrium.

For detailed proof see Appendix II. ■

Intuitively, β_{ij}^C can represent the largest influence level of community C_j to community C_i. When Υ_C is a P-matrix, usually it implies that β_{ij}^C for $\forall i,j \in \mathbb{N}[1,M], i \neq j$ has a relatively small value compared to 1. In this type of network, communities' actions have a bounded influence on each other. Where on the other hand, if at least one community's action has an outsized effect on other communities, its decision can shift the state of the network substantially and result in possibly multiple equilibriums.

It is worth noting that if we ignore the existence of community, we can form Υ^C by considering each agent as a singleton community. In this case, Υ^C would be an $N \times N$ matrix and would be equal to the matrix used by [23] for checking the uniqueness of Nash equilibrium. On the other hand, by taking advantage of the community structure, Υ^C is an $M \times M$ matrix which is smaller in dimension and easier to verify whether it is a P-matrix.

Let $\Gamma^C = \Upsilon^C - \boldsymbol{I}$ and $\rho(\Gamma^C)$ be the largest eigenvalue of Γ^C in magnitude ($\rho(\Gamma^C) = \lambda_{max}(\Gamma^C)$). By [24], if $\rho(\Gamma^C) < 1$, then Υ^C is a P-matrix. Therefore, by Proposition 2, we have the following.

Corollary 1: If $\rho(\Gamma^C) < 1$ and K is a Cartesian product $K = K_1 \times K_2 \times \cdots \times K_M$, then the network game has a unique Nash equilibrium.

By [25], we know that a symmetric matrix is a P-matrix iff it is positive definite. Moreover, if $\nabla F(\boldsymbol{x})$ is a symmetric matrix, then Υ^C and Γ^C both are symmetric. These facts lead us to the following corollary.

Corollary 2: Assume that Jacobian matrix $\nabla F(\boldsymbol{x})$ is a symmetric matrix on K, and K is a Cartesian product $K = K_1 \times K_2 \times \cdots \times K_M$. Moreover, let $\lambda_1(\Gamma^C) \leq \lambda_2(\Gamma^C) \leq \cdots \leq \lambda_M(\Gamma^C)$ be the eigenvalues of Γ^C. Then, the network game has a unique Nash equilibrium if,

1) Υ^C is positive definite, i.e., $\Upsilon^C \succ 0$, or,
2) eigenvalues of Γ^C are larger than -1, i.e., $\lambda_1(\Gamma^C) > -1$.

It is worth noting that the first and second conditions are equivalent.

Proof: $\Upsilon^C \succ 0$ implies that Υ^C is a P-matrix. By proposition (2), the Nash equilibrium is unique. Moreover,

$$\lambda_1(\Gamma^C) > -1 \Leftrightarrow \Gamma^C \succ -I \Leftrightarrow \Gamma^C + I = \Upsilon^C \succ 0 \quad (15)$$

As a result, if $\lambda_1(\Gamma^C) > -1$, the Nash equilibrium of the network game is unique. ■

Example 3: An example of a game with symmetric Jacobian matrix.

Consider a game with the following parameters:

$$N = 5, M = 3, C_1 = \{a_1^{(1)}, a_1^{(2)}\}, \ C_2 = \{a_2^{(1)}, a_2^{(2)}\}, \ C_3 = \{a_3^{(1)}\}$$

Assume that $f_1(\boldsymbol{x}) = \alpha W_{12} \cdot \boldsymbol{x}_2 + \alpha W_{13} \cdot \boldsymbol{x}_3$ and $f_2(\boldsymbol{x}) = \beta W_{21} \cdot \boldsymbol{x}_1 + \beta W_{23} \cdot \boldsymbol{x}_3$ and $f_3(\boldsymbol{x}) = \gamma W_{31} \cdot \boldsymbol{x}_1 + \gamma W_{32} \cdot \boldsymbol{x}_2$, where $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^2$ and $\boldsymbol{x}_3 \in \mathbb{R}$. Moreover, $\alpha, \beta, \gamma \in \mathbb{R} - \{0\}$ are three constants. We consider the following relation between edge weights,

$$W_{21} = \frac{\alpha}{\beta} \cdot W_{12}^T, \ W_{31} = \frac{\alpha}{\gamma} \cdot W_{13}^T, \ W_{32} = \frac{\beta}{\gamma} \cdot W_{23}^T$$

In this example, even though the adjacency matrix of the network is not symmetric, the Jacobian matrix $\nabla F(\boldsymbol{x})$ is symmetric and is given by,

$$\nabla F(\boldsymbol{x}) = \begin{bmatrix} I_{2\times 2} & -\alpha W_{12} & -\alpha W_{13} \\ -\alpha W_{12}^T & I_{2\times 2} & -\beta W_{23} \\ -\alpha W_{13}^T & -\beta W_{23}^T & 1 \end{bmatrix}$$

Again, we note that Γ^C could be formed by treating each agent as a singleton community, whereby Γ^C is an $N \times N$ matrix and Corollary 2 reduces to Proposition 3 of [13]. On the other hand, by using the community structure, Γ^C is an $M \times M$ matrix and conditions in Corollary 2 are easier to verify as compared to those in Proposition 3 of [13].

IV. Stability

We now examine the stability of Nash equilibrium in these games. When small changes occur to the underlying model parameters, a new Nash equilibrium may result. Intuitively, if the new Nash equilibrium is close enough to the original one, then we say the original Nash equilibrium is stable.

We define a family of parameterized interaction function $f_{i,k}(x,p) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, where p is a real number called *perturbation parameter* or *shock*. Let $p_i^{(k)}$ be the perturbation applied to agent $a_i^{(k)}$, $\boldsymbol{p}_i = [p_i^{(1)}, \cdots, p_i^{(N_i)}]$ the vector of perturbation applied to community i, and $\boldsymbol{p} = [\boldsymbol{p}_1, \cdots \boldsymbol{p}_M] \in \mathbb{R}^N$ the vector of all perturbations/shocks. Moreover, let $\boldsymbol{x}^*(\boldsymbol{p})$ be the action profile at the Nash equilibrium of the game under perturbation vector $\boldsymbol{p}$ and $\boldsymbol{x}^*$ be the Nash equilibrium of the unperturbed game ($\boldsymbol{x}^* = \boldsymbol{x}^*(\boldsymbol{0})$).

We denote a ball of radius $r > 0$ centered at $\boldsymbol{x} \in \mathbb{R}^N$ by $B(\boldsymbol{x}, r) := \{\boldsymbol{y} \in \mathbb{R}^N : ||\boldsymbol{x} - \boldsymbol{y}||_2 < r\}$.

Definition 4: ([26]) A Nash equilibrium $\boldsymbol{x}^*$ is stable if $\exists r > 0, d > 0$ such that $\forall \boldsymbol{p} \in B(\boldsymbol{0}, r)$, the Nash equilibrium $\boldsymbol{x}^*(\boldsymbol{p})$ exists and satisfies

$$||\boldsymbol{x}^*(\boldsymbol{p}) - \boldsymbol{x}^*||_2 \leq d||F(\boldsymbol{x}^*(\boldsymbol{p}), \boldsymbol{p}) - F(\boldsymbol{x}^*(\boldsymbol{p}), \boldsymbol{0})||_2 \ ,$$

where $F_i(\boldsymbol{x}, \boldsymbol{p}) = \boldsymbol{x}_i(\boldsymbol{p}) - f_i(\sum_{j=1}^M W_{ij}\boldsymbol{x}_j, \boldsymbol{p})$.

Definition 4 implies the Nash equilibrium of perturbed game remains close to the Nash equlibrium of unpurturbed game ($\boldsymbol{x}^*(0)$) if $\boldsymbol{x}^*(0)$ is stable.

A. Stability condition without community structure

In order to determine whether Nash equilibrium $\boldsymbol{x}^*$ is stable, [13] divides the agents into three disjoint sets based on $\boldsymbol{x}^*$:

$$\begin{aligned} A(\boldsymbol{x}^*) &:= \{a_i^{(k)} | x_i^{(k)*} > 0, x_i^{(k)*} - f_{i,k}([\sum_{j=1}^M W_{ij} \cdot \boldsymbol{x}_j^*]_k) = 0\}, \\ I(\boldsymbol{x}^*) &:= \{a_i^{(k)} | x_i^{(k)*} = 0, x_i^{(k)*} - f_{i,k}([\sum_{j=1}^M W_{ij} \cdot \boldsymbol{x}_j^*]_k) > 0\}, \\ B(\boldsymbol{x}^*) &:= \{a_i^{(k)} | x_i^{(k)*} = 0, x_i^{(k)*} - f_{i,k}([\sum_{j=1}^M W_{ij} \cdot \boldsymbol{x}_j^*]_k) = 0\} \ , \end{aligned}$$

where $A(\boldsymbol{x}^*)$ is referred to as the set of active agents, $I(\boldsymbol{x}^*)$ the set of strictly inactive agents and $B(\boldsymbol{x}^*)$ the set of borderline inactive agents. With small parametric perturbation $\boldsymbol{p}$, agents in $A(\boldsymbol{x}^*)$ remain active ($x_i^{(k)*}(\boldsymbol{p}) > 0$) and agents in I remain inactive ($x_i^{(k)*}(\boldsymbol{p}) = 0$), while agents in $B(\boldsymbol{x}^*)$ can transform from inactive to active.

[13] established the following sufficient condition for the solution to $VI(K, F)$ to be stable in the sense of Definition 4.

Theorem 2: ([13]) Consider the matrix

$$\nabla_{A,B} F_{A,B}(\boldsymbol{x}^*) = \begin{bmatrix} \nabla_A F_A(\boldsymbol{x}^*) & \nabla_B F_A(\boldsymbol{x}^*) \\ \nabla_A F_B(\boldsymbol{x}^*) & \nabla_B F_B(\boldsymbol{x}^*) \end{bmatrix} \tag{16}$$

where $\nabla_{S_1} F_{S_2}(\boldsymbol{x}^*)$ is a sub-matrix of $\nabla F(\boldsymbol{x}^*)$ whose rows and columns corresponding to the agents in S_1 and S_2, respectively, and $\nabla_{A,B} F_{A,B}(\boldsymbol{x}^*)$ is generated by selecting rows and columns corresponding to $A \cup B$ from the game Jacobian $\nabla F(\boldsymbol{x}^*)$. If $\nabla_{A,B} F_{A,B}(\boldsymbol{x}^*)$ is positive definite on K, then the solution $\boldsymbol{x}^*$ to $VI(K, F)$ is stable.

Next, we provide a condition for stability which is easier to verify as compared to that in Theorem 2 by taking community structure into account.

B. Stability condition with community structure

Similar to [13], we divide communities into active, strictly inactive and borderline inactive sets. Specifically, (1) a community is active if at least one agent in that community exerts nonzero effort at NE $\boldsymbol{x}^*$; (2) if all agents in a community are strictly inactive, then the community is strictly inactive; (3) if all agents of a community are inactive and at least one of them is borderline inactive, then the community is considered as a borderline inactive community. Formally, we have,

$$\begin{aligned} A_C(\boldsymbol{x}^*) &:= \{C_i | \boldsymbol{x}_i^* \neq \boldsymbol{0}\}, \\ I_C(\boldsymbol{x}^*) &:= \{C_i | \boldsymbol{x}_i^* = \boldsymbol{0}, \boldsymbol{x}_i^* - f_i(\sum_{j=1}^M W_{ij} \cdot \boldsymbol{x}_j^*) \succ \boldsymbol{0}\}, \\ B_C(\boldsymbol{x}^*) &:= \{C_i | \boldsymbol{x}_i^* = \boldsymbol{0}\} - I_C(\boldsymbol{x}^*), \end{aligned} \tag{17}$$

where $A_C(\boldsymbol{x}^*)$, $I_C(\boldsymbol{x}^*)$, $B_C(\boldsymbol{x}^*)$ denote the set of active, strictly inactive and borderline inactive communities respectively.

Proposition 3: Consider Nash equilibrium $\boldsymbol{x}^*$ in the game. Re-index all communities in $A_C(\boldsymbol{x}^*) \cup B_C(\boldsymbol{x}^*)$ using $1, 2, \cdots, Q$, $Q = |A_C(\boldsymbol{x}^*)| + |B_C(\boldsymbol{x}^*)|$, then define

$$G^C(\boldsymbol{x}^*) = \begin{bmatrix} 1 & -G_{12}^C(\boldsymbol{x}^*) & \dots & -G_{1Q}^C(\boldsymbol{x}^*) \\ -G_{21}^C(\boldsymbol{x}^*) & 1 & \dots & -G_{2Q}^C(\boldsymbol{x}^*) \\ \vdots & \vdots & \ddots & \vdots \\ -G_{Q1}^C(\boldsymbol{x}^*) & -G_{Q2}^C(\boldsymbol{x}^*) & \dots & 1 \end{bmatrix}$$

where $G_{ij}^C(\boldsymbol{x}^*) = ||\nabla_j F_i(\boldsymbol{x}^*)||_2$. If $G^C(\boldsymbol{x^*}) \succ \boldsymbol{0}$, then $\boldsymbol{x}^*$ is stable.

Proof: See Appendix III ■

Here $G^C(\boldsymbol{x}^*)$ captures the mutual influence between active and borderline inactive communities at the current effort profile. The borderline inactive communities can turn into active communities under parametric perturbations. When such flips are significant, large fluctuations can appear in the network, which can be further amplified through rebounds and reflections. In this case, new equilibria may not exist, and even if they do, they may be far away from the original equilibrium. But when $G^C(\boldsymbol{x}^*) \succ \boldsymbol{0}$ holds, the significance of flipping from inactive to active are bounded and fluctuations become small enough to ignore, and therefore the stability holds.

Matrix G^C is a $Q \times Q$ matrix. As $Q \leq M$, the dimension of G^C is smaller than the matrix $\nabla_{A,B} F_{A,B}(\boldsymbol{x}^*)$ defined in Theorem 2. Therefore, condition in Proposition 3 is easier to verify compared to Theorem 2.

We conclude this section by introducing a condition on Γ^C leading to stable Nash equilibrium.

Proposition 4: Assume ∇F is symmetric on K, and K is a Cartesian product $K = \prod_{i=1}^M K_i$. Moreover, Let $\lambda_1(\Gamma^C) \leq \lambda_2(\Gamma^C) \leq \cdots \leq \lambda_M(\Gamma^C)$ be the eigenvalues of Γ^C. If $\lambda_1(\Gamma^C) > -1$, then Nash equilibrium is unique and stable.

V. CENTRALITY

In network games, centrality is a metric to measure the influence of nodes on the network. Degree centrality is one of the centrality metrics which has gained attention in the literature [27], [28]. In a directed graph, two different measures of degree centrality are considered for each node; indegree centrality, which is a count of edges directed to a given node, and outdegree centrality, which is the number of outward directed edges from the given node.

In this section, we introduce a generalization of the degree centrality measure for disjoint communities.

We describe the influence of a community using Jacobian matrix $\nabla F(\boldsymbol{x})$. Matrix $\nabla_j F_i(\boldsymbol{x})$ measures the sensitivity of agents in community i to the effort of agents in community j. Thus $||\nabla_j F_i(\boldsymbol{x})||_2$ is an appropriate measure of influence of community j on community i. We formally define our generalized centrality measure as follows.

Definition 5: Generalized Degree Centrality (GDC): Let $\beta_{ij}^C = \sup_{\boldsymbol{x}\in K} ||\nabla_j F_i(\boldsymbol{x})||_2$. Then, the generalized degree centralities for community C_i are given by,

$$D_i^{in} = \sum_{j:j\neq i} \beta_{ij}^C,\ D_i^{out} = \sum_{j:j\neq i} \beta_{ji}^C,\ \forall i,j \in \mathbb{N}[1,M]$$

Moreover, the maximum GDCs are defined as follows,

$$D_{max}^{in} = \max_{i\in\mathbb{N}[1,M]} D_i^{in},\ D_{max}^{out} = \max_{i\in\mathbb{N}[1,M]} D_i^{out}$$

The above definition can be interpreted as follows: Outdegree centrality measures the influence of a given community on the network and depends on two factors, (1) number of links directed outward from the community, and (2) derivative of the best response function of other communities with respect to effort levels of the given community (Larger derivative implies the network is more sensitive to the given community's effort level). In Definition 5, D_i^{out} captures these two factor as it is the summation of $\beta_{ij}^C = \sup_{\boldsymbol{x}\in K} ||\nabla_j F_i(\boldsymbol{x})||_2$. We can make a similar argument for D_i^{in}.

If we wish to add external importance to the different communities, we can generalize the extended centrality measure defined in [29] as follows.

Definition 6: Generalized Extended Degree Centrality (GEDC):

Let $\boldsymbol{e} \in \mathbb{R}_{>0}^M$ denote the vector of external importance, where $(\boldsymbol{e})_i = e_i > 0$ shows the external importance we put on C_i. We denote

$$D_i^{in}(\boldsymbol{e}) = \sum_{j:j\neq i} \beta_{ij}^C \frac{e_j}{e_i},\ D_i^{out}(\boldsymbol{e}) = \sum_{j:j\neq i} \beta_{ji}^C \frac{e_i}{e_j}, \forall i,j \in \mathbb{N}[1,M]$$

as the generalized extended degree centralities for C_i, and

$$D_{max}^{in}(\boldsymbol{e}) = \max_{i\in\mathbb{N}[1,M]} D_i^{in}(\boldsymbol{e}), D_{max}^{out}(\boldsymbol{e}) = \max_{i\in\mathbb{N}[1,M]} D_i^{out}(\boldsymbol{e})$$

as the maximum GEDCs.

When $\boldsymbol{e} = \alpha\mathbf{1}, \alpha > 0$, Definition 5 and Definition 6 are equivalent. We now move on to the connection between our centrality measure and the uniqueness of Nash equilibrium.

Proposition 5: If there exists $\boldsymbol{e} \succ \mathbf{0}$ such that $D_{max}^{out}(\boldsymbol{e}) < 1$, then the Nash equilibrium is unique.

Proof: See Appendix IV. ∎

Similarly, we have

Proposition 6: If there exists $\boldsymbol{e} \succ \mathbf{0}$ such that $D_{max}^{in}(\boldsymbol{e}) < 1$, then the Nash equilibrium is unique.

The proof is similar to the proof of Proposition 5.

Proposition 5 and 6 both imply that if indegree or outdegree centrality is bounded, then the Nash equilibrium is unique. On the other hand, if neither the indegree nor outdegree is bounded, then at least one community has an outsized effect on the network. This community's decision can change the state of the network significantly, resulting in possibly multiple equilibrium.

Proposition 5 and 6 are similar with but different from Proposition 7 in [23]. In our work, the β_{ij}^C represents the influence level of communities on each other while β_{ij} in [23] represents the influence level of agents on each other. Moreover, when both conditions in Proposition 5 and 6 hold, $\Upsilon^C \succ 0$. and it becomes a special case of the condition (where Υ_C is symmetric) in Proposition 2.

If ∇F is symmetric on K, then we are able to establish the connection between our centrality measure and the stability of Nash equilibrium in the following.

Corollary 3: If ∇F is symmetric on K, and exists $\boldsymbol{e} = \alpha\mathbf{1},\ \alpha > 0$ such that $D_{max}^{out}(\boldsymbol{e}) = D_{max}^{in}(\boldsymbol{e}) < 1$, then the Nash equilibrium is unique and stable.

Proof: See Appendix V. ∎

Note that Corollary 3 implies Proposition 4 but not vice versa (See Appendix for more details).

Corollary 3 implies that if the degree centrality of the communities are bounded, then the network has a unique and stable Nash equilibrium. On the other hand, if the degree centrality of a given community is not bounded, the community has huge impact on the network and the Nash equilibrium may not be stable as a small perturbation can influence the network dramatically.

VI. Conclusion

We studied a family of games played on a network with the community structure and non-linear best response function. Prior works on network games have found sufficient conditions for uniqueness and stability of Nash equilibria which are mostly difficult to verify. In this work, we showed the existence of a community structure helps us to simplify such conditions. In particular, we show that the uniqueness and stability of Nash equlibria are related to matrices which are possibly lower dimensional with their dimensions depending on the number of communities. Moreover, we propose a new notion of degree centrality to evaluate the influence of a community in the network. Using this notion, we are able to identify conditions for uniqueness and stability. Specifically, we show that if communities of a network have a bounded degree centrality, then the corresponding network has a unique and stable Nash equilibrium.

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APPENDIX I
PROOF OF PROPOSITION 1

Proof: According to the finite budget assumptions in Proposition 1 part 1 and part 2, together with effort level lower bounded by 0, we know that K is always nonempty and compact. And according to the independence assumptions, K is a Cartesian product, in part 1,

$$K = \prod_{i=1}^{M} \prod_{k=1}^{N_i} K_i^{(k)},$$

and in part 2,

$$K = \prod_{i=1}^{M} K_i.$$

Since $K_i^{(k)}$ is convex in part 1 and K_i is convex in part 2, K is always a Cartesian product of convex sets and therefore is also convex. So K is nonempty, compact and convex, and therefore the Nash equilibrium exists following Theorem 1.

APPENDIX II
PROOF OF PROPOSITION 2

Proof: According to the assumption in Section II, F is continuously differentiable on K.

With the community level partition $K = \prod_{i=1}^{M} K_i$, we denote $\nabla F_i(\boldsymbol{z}) = ((\nabla_j F_i(\boldsymbol{z}))_{j=1}^{M})^T \in \mathbb{R}^{N_i \times N}$.

We use the notation $L(\boldsymbol{x},\boldsymbol{y})$ to denote the line segment between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^N$. Formally,

$$L(\boldsymbol{x},\boldsymbol{y}) = \{\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} : 0 \le \alpha \le 1\}.$$

According to our assumptions on F in Section II, $F_i : K \to \mathbb{R}^{N_i}, K \subseteq \mathbb{R}^N$ is continuously differentiable on K, and for $\forall \boldsymbol{x},\boldsymbol{y}$ in K_i, $L(\boldsymbol{x},\boldsymbol{y}) \subseteq K_i$. According to [30] page 355, Theorem 12.9, we know that for every vector $\boldsymbol{a}$ in $\mathbb{R}^{N_i}$, there is a point $\boldsymbol{z} \in L(\boldsymbol{x},\boldsymbol{y})$ such that:

$$\boldsymbol{a} \cdot (F_i(\boldsymbol{x}) - F_i(\boldsymbol{y})) = \boldsymbol{a} \cdot (\nabla F_i(\boldsymbol{z})(\boldsymbol{x}-\boldsymbol{y})). \quad (18)$$

Let $\boldsymbol{a}$ in equation (18) be $(\boldsymbol{x}_i-\boldsymbol{y}_i)^T$, and we denote $\boldsymbol{l}=(l_j)_{j=1}^M$, where $l_j=||\boldsymbol{x}_j-\boldsymbol{y}_j||_2, \forall j\in\mathbb{N}[1,M]$, then,

$$\begin{aligned}
& (\boldsymbol{x}_i-\boldsymbol{y}_i)^T(F_i(\boldsymbol{x})-F_i(\boldsymbol{y})) \\
= \quad & (\boldsymbol{x}_i-\boldsymbol{y}_i)^T(\nabla F_i(\boldsymbol{z})(\boldsymbol{x}-\boldsymbol{y})) \\
= \quad & (\boldsymbol{x}_i-\boldsymbol{y}_i)^T[\sum_{j=1}^M \nabla_j F_i(\boldsymbol{z})(\boldsymbol{x}_j-\boldsymbol{y}_j)] \\
\geq \quad & (\boldsymbol{x}_i-\boldsymbol{y}_i)^T \nabla_i F_i(\boldsymbol{z})(\boldsymbol{x}_i-\boldsymbol{y}_i) \\
& -|\sum_{j\neq i}(\boldsymbol{x}_i-\boldsymbol{y}_i)^T\nabla_j F_i(\boldsymbol{z})(\boldsymbol{x}_j-\boldsymbol{y}_j)| \\
\geq \quad & (l_i)^2-\sum_{j\neq i}\beta_{ij}^C\cdot l_i\cdot l_j \\
= \quad & l_i\cdot(\Upsilon^C\boldsymbol{l})_i
\end{aligned} \tag{19}$$

In [31] Theorem 3.3.4(b), a real square matrix $M\in\mathbb{R}^{n\times n}$ is a P-matrix if it satisfies

$$l_i(M\boldsymbol{l})_i>0, \forall i\in\mathbb{R}^n$$

which is equivalent to

$$\min_{i\in\mathbb{N}[1,M]} l_i(M\boldsymbol{l})_i>0$$

Denote $b=\min_{i\in\mathbb{N}[1,M]}\frac{l_i\cdot[\Upsilon^C\boldsymbol{l}]_i}{||\boldsymbol{l}||_2^2}>0$ we have

$$\max_{i\in\mathbb{N}[1,M]}(\boldsymbol{x}_i-\boldsymbol{y}_i)^T(F_i(\boldsymbol{x})-F_i(\boldsymbol{y}))\geq l_i\cdot[\Upsilon^C\boldsymbol{l}]_i\geq b\cdot||\boldsymbol{x}-\boldsymbol{y}||_2^2 \tag{20}$$

which, according to Definition 2.(b), shows that F satisfies uniform block P-condition. And based on [12] Proposition 2 part (b) and [32] Proposition 3.5.10 part (b), we know that the Nash equilibrium is unique.

Appendix III
Proof of Proposition 3

Denote

$$S=\{a_k|a_k\in C_i, s.t. C_i\in A_C(\boldsymbol{x}^*)\cup B_C(\boldsymbol{x}^*)\}$$

as the set of all agents that belong to communities in $A_C(\boldsymbol{x}^*)\cup B_C(\boldsymbol{x}^*)$. Then similar to theorem 2, $\nabla_S F_S(\boldsymbol{x}^*)$ is a sub-matrix of $\nabla F(\boldsymbol{x}^*)$ whose columns and rows correspond to the agents in S. For $\forall\boldsymbol{y}\in\mathbb{R}_{\geq 0}^{|S|}$, we denote $\boldsymbol{l}=(l_i)_{i=1}^Q\in\mathbb{R}_{\geq 0}^Q$, where $l_i=||\boldsymbol{y_i}||_2, \forall i\in\mathbb{N}[1,Q]$. Then we have

$$\begin{aligned}
& \boldsymbol{y}^T\nabla_S F_S(\boldsymbol{x}^*)\boldsymbol{y} \\
= \quad & \sum_{i=1}^Q\sum_{j=1}^Q \boldsymbol{y_i}^T(\nabla_j F_i(\boldsymbol{x}^*))\boldsymbol{y_j} \\
= \quad & \sum_{i=1}^Q \boldsymbol{y_i}^T\boldsymbol{y_i}+\sum_{i=1}^Q\sum_{j=1,j\neq i}^Q \boldsymbol{y_i}^T(\nabla_j F_i(\boldsymbol{x}^*))\boldsymbol{y_j} \\
\geq \quad & \sum_{i=1}^Q||\boldsymbol{y_i}||_2^2-\sum_{i=1}^Q\sum_{j=1,j\neq i}^Q G_{ij}^C(\boldsymbol{x}^*)\cdot||\boldsymbol{y_i}||_2\cdot||\boldsymbol{y_j}||_2 \\
= \quad & \sum_{i=1}^Q {l_i}^2-\sum_{i=1}^Q\sum_{j=1,j\neq i}^Q G_{ij}^C(\boldsymbol{x}^*)\cdot l_i\cdot l_j \\
= \quad & \boldsymbol{l}^T G^C(\boldsymbol{x}^*)\boldsymbol{l} \ > \ 0
\end{aligned} \tag{21}$$

which shows $\nabla_S F_S(\boldsymbol{x}^*)\succ\boldsymbol{0}$. According to equation (16) and equation (17), we know that any agent in $A(\boldsymbol{x}^*)\cup B(\boldsymbol{x}^*)$ is in S. Therefore, $\nabla_{A,B}F_{A,B}(\boldsymbol{x}^*)$ is a principal minor of $\nabla_S F_S(\boldsymbol{x}^*)$, which means $\nabla_{A,B}F_{A,B}(\boldsymbol{x}^*)\succ\boldsymbol{0}$. Then according to theorem 2, the Nash equilibrium is stable.

Appendix IV
Proof of Proposition 5

We will first introduce the following definitions and theorems.

Definition 7: Equivalent norms:

On a vector space K, two norms g and h are equivalent if there exist two constants $c>0, C>0$ such that for $\forall\boldsymbol{x}\in K$, $c\cdot h(\boldsymbol{x})\leq g(\boldsymbol{x})\leq C\cdot h(\boldsymbol{x})$

Definition 8: For $\forall\boldsymbol{x}\in\mathbb{R}_{\geq 0}^N$, partition on $\boldsymbol{x}$, such that

$$\boldsymbol{x}=(\boldsymbol{x_i})_{i=1}^M,\ \boldsymbol{x_i}\in\mathbb{R}_{\geq 0}^{N_i},\ \sum_{i=1}^M N_i=N.$$

For $\forall\boldsymbol{v}=(v_i)_{i=1}^M\in\mathbb{R}_{>0}^M$, define norm $||\boldsymbol{x}||_{2,\boldsymbol{v}}$

$$||\boldsymbol{x}||_{2,\boldsymbol{v}}=\sum_{i=1}^M v_i\cdot||\boldsymbol{x_i}||_2 \tag{22}$$

$||\boldsymbol{x}||_{2,\boldsymbol{v}}$ and $||\boldsymbol{x}||_2$ are equivalent norms on $\mathbb{R}_{\geq 0}^N$ because $v_{min}\cdot||\boldsymbol{x}||_2\leq||\boldsymbol{x}||_{2,\boldsymbol{v}}\leq v_{max}\cdot\sqrt{M}\cdot||\boldsymbol{x}||_2$, where $v_{min}=\min_{i\in\mathbb{N}[1,M]}v_i$, $v_{max}=\max_{i\in\mathbb{N}[1,M]}v_i$.

Definition 9: Contraction Mapping:

In a metric space (K,d), a map $f:K\rightarrow K$ is called a contraction mapping on K if there exists $q\in[0,1)$ such that $d(f(\boldsymbol{x}),f(\boldsymbol{y}))\leq qd(\boldsymbol{x},\boldsymbol{y}),\ \forall\boldsymbol{x},\boldsymbol{y}\in K$.

Theorem 3: Banach fixed-point theorem:

Let (K,d) be a non-empty complete metric space with a contraction mapping $f:K\rightarrow K$. Then f admits a unique fixed-point $\boldsymbol{x}^*$ ($f(\boldsymbol{x}^*)=\boldsymbol{x}^*$) in K.

We denote the best response using the a mapping function $\tilde{f}(\boldsymbol{x})=(\tilde{f}_i(\boldsymbol{x}))_{i=1}^M$, where

$$\tilde{f}_i(\boldsymbol{x})=\max\{f_i(\sum_{j=1}^M W_{ij}\cdot\boldsymbol{x}_j),\boldsymbol{0}\},\ \forall i\in\{1,2,\cdots,M\}$$

Then we have $\tilde{f}(\boldsymbol{x})$ continuously differentiable on K, and $\nabla_i\tilde{f}_i(\boldsymbol{x})=\boldsymbol{0},\ ||\nabla_j\tilde{f}_i(\boldsymbol{x})||_2\leq\beta_{ij}^C$.

Then consider the metric space $(K,||\cdot||_{2,\boldsymbol{v}})$ and the contraction mapping $\tilde{f}:K\rightarrow K$, we show the following proof.

Proof: We Denote $\nabla\tilde{f}_i(\boldsymbol{z})=((\nabla_j\tilde{f}_i(\boldsymbol{z}))_{j=1}^M)^T\in\mathbb{R}^{N_i\times N}$, which is continuously differentiable on K. Similar to Appendix II, based on the equation (18) of Theorem 12.9 in [30] page 355, we know for every vector $\boldsymbol{a}\in\mathbb{R}^{N_i}$, there exists a point $\boldsymbol{z}\in L(\boldsymbol{x},\boldsymbol{y})$ such that:

$$\boldsymbol{a}\cdot(\tilde{f}_i(\boldsymbol{x})-\tilde{f}_i(\boldsymbol{y}))=\boldsymbol{a}\cdot(\nabla\tilde{f}_i(\boldsymbol{z})(\boldsymbol{x}-\boldsymbol{y})),\ \forall i\in\mathbb{N}[1,M].$$

According to the Cauchy Schwarz inequality, we further have,

$$||\tilde{f}_i(\boldsymbol{x}) - \tilde{f}_i(\boldsymbol{y})||_2 \leq ||\nabla \tilde{f}_i(\boldsymbol{z}) \cdot (\boldsymbol{x}-\boldsymbol{y})||_2, \ \forall i \in \mathbb{N}[1,M] \quad (23)$$

which can also be written as

$$||\tilde{f}_i(\boldsymbol{x}) - \tilde{f}_i(\boldsymbol{y})||_2 \leq ||\sum_{j=1}^{M} \nabla_j \tilde{f}_i(\boldsymbol{z}) \cdot (\boldsymbol{x}_j - \boldsymbol{y}_j)||_2.$$

Then with $\boldsymbol{z}$, let $\boldsymbol{v} = (\frac{1}{e_i})_{i=1}^{M} \succ \boldsymbol{0}$

$$\begin{aligned}
& ||\tilde{f}(\boldsymbol{x}) - \tilde{f}(\boldsymbol{y})||_{2,\boldsymbol{v}} \\
= \ & \sum_{i=1}^{M} \frac{1}{e_i} \cdot ||\tilde{f}_i(\boldsymbol{x}) - \tilde{f}_i(\boldsymbol{y})||_2 \\
\leq \ & \sum_{i=1}^{M} \frac{1}{e_i} \cdot ||\nabla \tilde{f}_i(\boldsymbol{z}) \cdot (\boldsymbol{x}-\boldsymbol{y})||_2 \\
= \ & \sum_{i=1}^{M} \frac{1}{e_i} \cdot ||\sum_{j=1}^{M} \nabla_j \tilde{f}_i(\boldsymbol{z}) \cdot (\boldsymbol{x}_j - \boldsymbol{y}_j)||_2 \\
\leq \ & \sum_{i=1}^{M} \sum_{j=1}^{M} \frac{1}{e_i} \cdot ||\nabla_j \tilde{f}_i(\boldsymbol{z}) \cdot (\boldsymbol{x}_j - \boldsymbol{y}_j)||_2 \\
\leq \ & \sum_{i=1}^{M} \sum_{j=1}^{M} \frac{1}{e_i} \cdot ||\nabla_j \tilde{f}_i(\boldsymbol{z})||_2 \cdot ||\boldsymbol{x}_j - \boldsymbol{y}_j||_2 \\
\leq \ & \sum_{i=1}^{M} \sum_{j=1, j\neq i}^{M} \frac{1}{e_i} \cdot \beta_{ij}^{C} \cdot ||\boldsymbol{x}_j - \boldsymbol{y}_j||_2 \\
= \ & \sum_{i=1}^{M} \sum_{j=1, j\neq i}^{M} \frac{1}{e_i} \cdot \frac{e_j}{e_j} \cdot \beta_{ij}^{C} \cdot ||\boldsymbol{x}_j - \boldsymbol{y}_j||_2 \\
= \ & \sum_{i=1}^{M} \sum_{j=1, j\neq i}^{M} \frac{1}{e_j} \cdot ||\boldsymbol{x}_j - \boldsymbol{y}_j||_2 \cdot \beta_{ij}^{C} \cdot \frac{e_j}{e_i} \\
= \ & \sum_{j=1}^{M} \frac{1}{e_j} \cdot ||\boldsymbol{x}_j - \boldsymbol{y}_j||_2 \cdot \sum_{i=1, i\neq j}^{M} \beta_{ij}^{C} \cdot \frac{e_j}{e_i} \\
\leq \ & ||\boldsymbol{x} - \boldsymbol{y}||_{2,\boldsymbol{v}} \cdot D_{max}^{out}(\boldsymbol{e}) \qquad (24)
\end{aligned}$$

Then with the Banach fixed-point theorem, we know that a unique Nash equilibrium exists in the network game.

APPENDIX V
PROOF OF COROLLARY 3

Proof: The uniqueness is obvious from Proposition 5. Denote $D_{max} = D_{max}^{out}(\boldsymbol{e}) = \max_{i \in \mathbb{N}[1,M]} \sum_{j: j\neq i} \beta_{ji}^{C}$, and $l_i = ||\boldsymbol{x_i}||_2, \forall i \in \mathbb{N}[1,M]$ then for $\forall \boldsymbol{x} \in \mathbb{R}_{\geq 0}^{N}$, $\boldsymbol{x} \neq \boldsymbol{0}$ we have

$$\begin{aligned}
& \boldsymbol{x}^T \nabla F(\boldsymbol{x}) \boldsymbol{x} \\
\geq \ & \sum_{i=1}^{M} \sum_{j=1}^{M} \boldsymbol{x_i}^T (\boldsymbol{I} + \Gamma^C)_{ij} \boldsymbol{x_j} \\
= \ & \sum_{i=1}^{M} \boldsymbol{x_i}^T \boldsymbol{x_i} + \sum_{i=1}^{M} \sum_{j=1, j\neq i}^{M} \boldsymbol{x_i}^T (\Gamma^C)_{ij} \boldsymbol{x_j} \\
\geq \ & \sum_{i=1}^{M} ||\boldsymbol{x_i}||_2^2 - \sum_{i=1}^{M} \sum_{j=1, j\neq i}^{M} \beta_{ij}^{C} \cdot ||\boldsymbol{x_i}||_2 \cdot ||\boldsymbol{x_j}||_2 \\
= \ & \sum_{i=1}^{M} {l_i}^2 - 2 \cdot \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \beta_{ij}^{C} \cdot l_i \cdot l_j \\
= \ & \sum_{i=1}^{M} (1 - \sum_{j=1, j\neq i}^{M} \beta_{ji}^{C}) \cdot {l_i}^2 + \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \beta_{ij}^{C} \cdot (l_i - l_j)^2 \\
\geq \ & \sum_{i=1}^{M} (1 - D_{max}) {l_i}^2 + \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \beta_{ij}^{C} \cdot (l_i - l_j)^2 > 0
\end{aligned} \qquad (25)$$

so $\nabla F(\boldsymbol{x}) \succ \boldsymbol{0}$ on K. We denote the unique Nash equilibrium as $\boldsymbol{x}^*$, then $\nabla F(\boldsymbol{x}^*) \succ \boldsymbol{0}$ and its principal minor $\nabla_{A,B} F_{A,B}(\boldsymbol{x}^*) \succ \boldsymbol{0}$. From Theorem 3, the Nash equilibrium is stable.